J. Fluid Mech. (1979), *vol.* 90, *part* 1, *pp.* 161–178
Printed in Great Britain

On resonant over-reflexion of internal gravity waves from a Helmholtz velocity profile

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(Received 2 November 1977)

A Helmholtz velocity profile with velocity discontinuity $2U$ is embedded in an infinite continuously stratified Boussinesq fluid with constant Brunt–Väisälä frequency N. Linear theory shows that this system can support resonant over-reflexion, i.e. the existence of neutral modes consisting of outgoing internal gravity waves, whenever the horizontal wavenumber is less than $N/2^{\frac{1}{2}}U$. This paper examines the weakly nonlinear theory of these modes. An equation governing the evolution of the amplitude of the interface displacement is derived. The time scale for this evolution is α^{-2}, where α is a measure of the magnitude of the interface displacement, which is excited by an incident wave of magnitude $O(\alpha^3)$. It is shown that the mode which is symmetrical with respect to the interface (and has a horizontal phase speed equal to the mean of the basic velocity discontinuity) remains neutral, with a finite amplitude wave on the interface. However, the other modes, which are not symmetrical with respect to the interface, become unstable owing to the self-interaction of the primary mode with its second harmonic. The interface displacement develops a singularity in a finite time.

1. Introduction

In a recent paper Acheson (1976) reviewed the phenomena of over-reflexion, in which a wave incident upon a shear layer generates a reflected wave of greater magnitude, as well as a transmitted wave. He examined the energetic aspects of the phenomena, and described the way in which the excess reflected energy is extracted from the mean motion. A special case of over-reflexion is resonant over-reflexion when, according to linear theory, there is no incident wave and the shear layer spontaneously emits only outgoing waves. Earlier, Lindzen (1974) had drawn attention to this phenomenon in his study of the stability of a Helmholtz velocity profile embedded in an infinite continuously stratified Boussinesq fluid. Lindzen was motivated by some observations (Ottersten, Hardy & Little 1973) which showed large amplitude internal gravity waves in the neighbourhood of shear layers. Since clear-air turbulence is generally attributed to the instability of shear layers (see Atlas *et al.* 1970; or the review by Dutton & Panofsky 1970), Lindzen wished to investigate the possibility that the energy flux associated with outgoing internal gravity waves may inhibit instability of the shear layer.

For the Helmholtz velocity profile with a basic velocity discontinuity of $2U$ and Brunt–Väisälä frequency N (figure 1), perturbations with horizontal wavenumbers k are unstable when $k > N/2^{\frac{1}{2}}U$ (Drazin & Howard 1966, p. 46, where references to earlier work on this model are given). For $k < N/U$, however, there also exists a

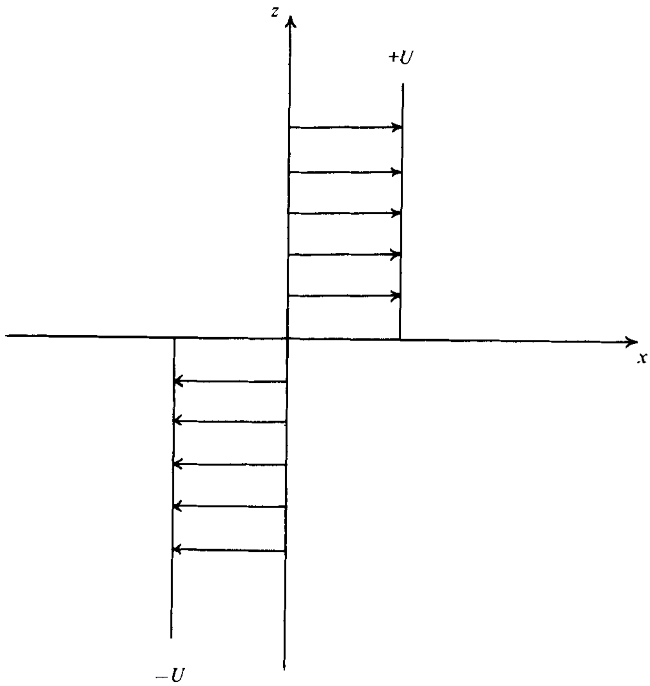

FIGURE 1. The basic velocity profile and co-ordinate system.

neutral mode, consisting of outgoing internal gravity waves; this mode is an example of resonant over-reflexion. Lindzen's discussion of this phenomenon was confined to linear theory, and his conclusions were based solely on the wave energy flux associated with the outgoing waves *vis-à-vis* the growth in energy of the unstable modes. However, Acheson (1976) showed that the energetics of resonant over-reflexion require a calculation of the mean flows generated by the outgoing wave packets, and the consequent changes in the total energy budget associated with these mean flows. McIntyre & Weissman (1978) have also drawn attention to these mean flows in a general discussion of radiating instabilities and resonant over-reflexion. Grimshaw (1976) undertook a weakly nonlinear analysis of the interaction between the unstable modes and the neutral mode for wavenumbers close to the critical wavenumber $N/2^{\frac{1}{2}}U$, and showed that the effect of the self-interaction of the second harmonic and the mean flow with the primary mode was the ultimate development of a finite amplitude wave on the interface (where the basic profile is discontinuous).

For $N/2U < k < N/2^{\frac{1}{2}}U$ the model contains three neutral modes, each consisting of outgoing internal gravity waves and exhibiting the phenomena of resonant over-reflexion. One of these modes exists for $0 < k < N/U$ and has a horizontal phase velocity equal to the mean of the basic velocity discontinuity (zero in the frame of reference described in figure 1); this mode (called mode (i) hereafter) is symmetrical with respect to the interface. The other two modes have equal but opposite phase velocities, are not symmetrical with respect to the interface and will be designated as mode (ii) hereafter. Lindzen (1974) suggested that the existence of resonant over-reflexion would imply instability if a lower boundary were added to the model; calculations by Lindzen & Rosenthal (1976) confirm this, using linear stability theory.

In this paper we shall examine the weakly nonlinear aspects of resonant over-reflexion. It will be shown that mode (i) remains neutrally stable, with a finite amplitude wave on the interface, while mode (ii) is unstable owing to the self-interaction of the second harmonic with the primary mode. These results suggest that finite amplitude internal gravity waves may co-exist with shear layers provided that their horizontal phase velocity is equal to the mean of the basic velocity discontinuity across the shear layer [i.e. waves corresponding to mode (i)]. Waves corresponding to mode (ii) may also be observed, but they will have limited lifetimes.

We shall assume that the basic state, in an infinite, inviscid, Boussinesq fluid, has a constant Brunt–Väisälä frequency N and a velocity in the x direction of $\pm U$ in $z \gtrless 0$ (figure 1). It will be assumed there is no variation in the y direction, as it may be shown that the stability criterion is independent of the wavenumber in the y direction. We shall use non-dimensional variables, based on a velocity scale U, a time scale N^{-1} and a length scale UN^{-1}; the reduced pressure (i.e. the deviation of the pressure from its hydrostatic value) is scaled by $\rho_1 U^2$, where ρ_1 is a reference density. Then the equations of motion are (e.g. Turner 1973, chap. 1)

$$u_x + w_z = 0, \tag{1.1a}$$

$$\pm u_x + u_t + p_x = F_H = -uu_x - wu_z, \tag{1.1b}$$

$$\pm w_x + w_t + p_z = F_V = -uw_x - ww_z, \tag{1.1c}$$

$$\pm r_x + r_t - w = G = -ur_x - wr_z. \tag{1.1d}$$

Here u and w are the x and z components of the perturbed velocity, p is the reduced pressure and r is the buoyancy (i.e. $g(\rho - \rho_0)/\rho_0$ scaled by UN, where $\rho_0(z)$ is the density in the basic state). The equations have been written in a form in which the linear terms are on the left-hand side and the nonlinear terms, represented by F_H, F_V and G, are on the right-hand side. The symbols $\pm$ indicate the regions $z \gtrless \zeta$, where $z = \zeta$ is the equation of the perturbed interface. If the variables on the left-hand side are eliminated in favour of w, it follows that

$$L^{\pm} w = M^{\pm}, \tag{1.2a}$$

where
$$L^{\pm}\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\} = -\left\{\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right\}^2 \left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right\} - \frac{\partial^2}{\partial x^2}, \tag{1.2b}$$

and
$$M^{\pm} = \frac{\partial^2}{\partial x^2}\left\{G - \frac{\partial F_V}{\partial t} \mp \frac{\partial F_V}{\partial x}\right\} + \frac{\partial^2}{\partial x \partial z}\left\{\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x}\right\} F_H. \tag{1.2c}$$

Here $L^{\pm}$ is a *linear* operator and $M^{\pm}$ contains the nonlinear terms.

The boundary conditions at the interface $z = \zeta$ are continuity of the pressure and the kinematic condition

$$\zeta_t \pm \zeta_x + u\zeta_x - w = 0 \quad \text{at} \quad z = \zeta. \tag{1.3}$$

Anticipating that ζ will be small, we expand this condition in a Taylor series about $z = 0$, so that

$$\zeta_t \pm \zeta_x - w = H^{\pm} \quad \text{at} \quad z = 0\pm, \tag{1.4a}$$

where
$$H^{\pm} = -(u\zeta)_x - (\tfrac{1}{2}\zeta^2 u_z)_x - \dots \quad \text{at} \quad z = 0\pm. \tag{1.4b}$$

Here we have used (1.1*a*) to simplify the nonlinear terms $H^{\pm}$. Similarly, the pressure condition at the interface is

$$[p]^{+}_{-} = Q, \tag{1.5a}$$

where

$$Q = -\zeta[p_z]^{+}_{-} - \tfrac{1}{2}\zeta[p_{zz}]^{+}_{-} - \dots. \tag{1.5b}$$

Here $[p]^{+}_{-}$ etc. denote the discontinuities in $[p]$ etc. across $z = 0$.

The linearized equations, which were discussed by Drazin & Howard (1966), Lindzen (1974) and Lindzen & Rosenthal (1976), are now obtained by formally putting F_H, F_V, G, $H^{\pm}$ and Q equal to zero. Seeking solutions of (1.1) proportional to $\exp\{ik(x-ct)\}$, we find that

$$w = \alpha A^{\pm} \exp\{ik(x-ct) \pm in^{\pm}z\} + \alpha_0 I^{\pm} \exp\{ik(x-ct) \mp in^{\pm}z\} \quad \text{in} \quad z \gtrless 0, \tag{1.6a}$$

$$\zeta = \alpha A \exp\{ik(x-ct)\}. \tag{1.6b}$$

Here A, $A^{\pm}$ and $I^{\pm}$ are constant amplitudes, α is a small parameter introduced as an appropriate measure of the magnitude of ζ and of the reflected waves (i.e. those terms associated with $A^{\pm}$), while α_0 is a small parameter which measures the magnitude of the incident waves (i.e. those terms associated with $I^{\pm}$). We shall assume thoughout that k is positive. The constants $n^{\pm}$ are given by

$$(n^{\pm})^2 = (c \mp 1)^{-2} - k^2, \tag{1.7}$$

and the appropriate branch is selected by applying a radiation condition. This condition is derived here by requiring the exponent of the reflected wave, $\exp\{\pm in^{\pm}z\}$, to decay exponentially when c_i (the imaginary part of c) takes small positive values. Lighthill (1960) has shown how this condition is derived by considering an appropriate initial-value problem and is equivalent to radiation conditions based on group-velocity criteria. If $n^{\pm}$ is $n_r^{\pm} + in_i^{\pm}$ and c is $c_r + ic_i$, then the radiation condition is

$$n_i^{\pm} > 0, \quad \text{or} \quad n_i^{\pm} = 0, \quad n_r^{\pm}(c_r \mp 1) < 0. \tag{1.8}$$

Substituting (1.6) into the linearized boundary conditions (1.4) and (1.5), it follows that

$$\alpha A^{\pm} + \alpha_0 I^{\pm} = -ik(c \mp 1)\,\alpha A \tag{1.9a}$$

and

$$n^{+}(c-1)\,\alpha A^{+} + n^{-}(c+1)\,\alpha A^{-} = n^{+}(c-1)\,\alpha_0 I^{+} + n^{-}(c+1)\,\alpha_0 I^{-}. \tag{1.9b}$$

For resonant over-reflexion these equations have a non-trivial solution for A when $I^{\pm}$ are both zero. The condition for this to be so is

$$n^{+}(c-1)^2 + n^{-}(c+1)^2 = 0. \tag{1.10}$$

This is a dispersion relation which determines c as a function of k (q.v. Drazin & Howard 1966, p. 46). The solutions are

$$\text{(i)} \quad c = 0 \quad \text{for} \quad 0 < k^2 < 1, \tag{1.11a}$$

or

$$\text{(ii)} \quad c^2 = (2k^2)^{-1} - 1 \quad \text{for} \quad k^2 > \tfrac{1}{4}. \tag{1.11b}$$

The solution (i) represents resonant over-reflexion of internal gravity waves (stationary in the present frame of reference); the restriction on k is obtained from the radiation condition (1.8). The solution (ii) also represents resonant over-reflexion for $\frac{1}{4} < k^2 < \frac{1}{2}$; the lower bound on k is obtained from the radiation condition (1.8) and implies that the phase speed c is bounded by unity. For $k^2 > \frac{1}{2}$, the solution (ii) is an unstable mode. The corresponding values of $n^{\pm}$ are

$$\text{(i)}\quad n^{\pm} = \pm(1-k^2)^{\frac{1}{2}}, \tag{1.12a}$$

$$\text{(ii)}\quad n^{\pm} = \pm k(1\pm c)/(1\mp c). \tag{1.12b}$$

In the nonlinear analysis that follows we shall consider wavenumbers k such that $\frac{1}{2} < k < 2^{-\frac{1}{2}}$. We let ϵ be a small parameter, put

$$T = \epsilon t, \quad Z = \epsilon z, \tag{1.13}$$

and allow the amplitude A to depend on the slow time scale T, while $A^{\pm}$ (and $I^{\pm}$) depend on both T and Z. This is a familiar technique in weakly nonlinear calculations. The form of the amplitude equation, and the appropriate scaling, may be deduced by the following heuristic argument. The introduction of the slow time variable T implies that the time derivative $\partial/\partial t$ is $-ikc+\epsilon\,\partial/\partial T$. Thus the linear part of the amplitude equation may be found by replacing kc in (1.9) by

$$\omega = kc + i\epsilon\,\partial/\partial T \tag{1.14}$$

and interpreting the result operationally. Hence the amplitude equation may be expected to be

$$\alpha\mathscr{D}(\omega,k)\,A = \alpha_0\{\mathscr{D}^{+}(\omega,k)\,I^{+} + \mathscr{D}^{-}(\omega,k)\,I^{-}\} + J, \tag{1.15}$$

where $\mathscr{D}(kc,k)$ is proportional to the dispersion relation [the left-hand side of (1.10)], $\mathscr{D}^{\pm}(kc,k)$ are non-zero constants, $I^{\pm}$ are evaluated at $Z = 0\pm$ and J represents the nonlinear terms. In § 2 the weakly nonlinear theory is described and precise expressions obtained for $\mathscr{D}(\omega,k)$, $\mathscr{D}^{\pm}(\omega,k)$ and J [q.v. (2.12)–(2.14)]. Here it suffices to observe that $\mathscr{D}(kc,k)$ vanishes, and so the leading term on the left-hand side of (1.15) is

$$i\epsilon\,\alpha\mathscr{D}_{\omega}(kc,k)\,\partial A/\partial T. \tag{1.16}$$

Further, it may be anticipated that the nonlinear term J will be proportional to $\alpha^3|A|^2A$. Hence the required balance between the time-evolution term, the forcing term and the nonlinear term demands that $\epsilon = \alpha^2$ and $\alpha_0 = \alpha^3$. The amplitude equation is

$$\partial A/\partial T = \beta|A|^2A + I, \tag{1.17}$$

where the known forcing term I is a linear combination of $I^{\pm}$ (evaluated at $Z = 0\pm$) and represents the effect of the incident wave packets on the interface. Away from the interface in $z \gtrless 0$, the solution is described (to leading order in α) by (1.6), where $A^{\pm}$ at $Z = 0\pm$ are determined in terms of A by (1.9a) and modulations in $|A^{\pm}|$ propagate vertically upwards in $z > 0$ (downwards in $z < 0$) at the vertical group velocity (q.v. (2.6) and the subsequent discussion).

The calculation of the nonlinear term J and the coefficient β is the principal purpose of this paper. The procedure and the results are described briefly in § 2; the full analysis is very long and is given elsewhere (Grimshaw 1977). In § 3 some solutions of the amplitude equation (1.17) are discussed, and in § 4 there is a summary of the main conclusions.

For mode (i) it transpires that β is zero and J is $O(\alpha^5)$. Nevertheless we shall retain the choice $\epsilon = \alpha^2$ and $\alpha_0 = \alpha^3$ for this case as the nonlinear terms in the equations (1.1) for the behaviour away from the interface are non-trivially $O(\alpha^3)$ [i.e. the terms $N_1^{\pm}$

in (2.6)]. The evolution of the interface amplitude A is determined by the forcing term I only for times $O(\alpha^{-2})$. A complete determination of the evolution of A including nonlinear effects would require the use of a longer time scale (say $\alpha^4 t$) and will not be considered in this paper. For mode (ii) it transpires that the real part of β [q.v. (5.2)] is positive, owing principally to the trapped nature of the second harmonic. This fact, and its implications for the solution of (1.17), is fully discussed in § 3.

When $k < \frac{1}{2}$, mode (ii) ceases to exist. However, for mode (i) and this range of k, both the primary wave and the second harmonic satisfy the dispersion relation (1.10) so there is a resonance between these two. This is discussed briefly at the end of § 4. On the other hand, when k is close to the critical wavenumber $2^{-\frac{1}{2}}$ modes (i) and (ii) coalesce. This was discussed fully by Grimshaw (1976). At the end of § 4 we complement that paper by discussing briefly the effect of a weak forcing. Further details of both these problems are given by Grimshaw (1977).

2. Weakly nonlinear theory

Motivated by the discussion at the end of the last section, we are led to consider expansions of the form

$$\zeta = \sum_{m=-\infty}^{\infty} \zeta_m(T) \exp\{imk(x-ct)\}, \tag{2.1a}$$

$$w = \sum_{m=-\infty}^{\infty} w_m(T,z,Z) \exp\{imk(x-ct)\}. \tag{2.1b}$$

Here $\zeta_{-m} = \bar{\zeta}_m$, $w_{-m} = \bar{w}_m$, $\zeta_1(T) = \alpha A(T)$ and c satisfies the dispersion relation (1.11) for resonant over-reflexion. These expressions and the corresponding expressions for u, r and p are substituted into (1.2) and the boundary conditions (1.4) and (1.5). The result is, on equating like Fourier components,

$$L^{\pm}\left\{-imkc+\epsilon\frac{\partial}{\partial T}, imk, \frac{\partial}{\partial z}+\epsilon\frac{\partial}{\partial Z}\right\} w_m = M_m^{\pm} \quad \text{in} \quad z \gtrless 0, \tag{2.2a}$$

$$-imk(c\mp 1)\,\zeta_m+\epsilon\,\partial\zeta_m/\partial T - w_m = H_m^{\pm} \quad \text{at} \quad z = 0\pm, \tag{2.2b}$$

$$[p_m]_-^+ = Q_m \quad \text{at} \quad z = 0\pm. \tag{2.2c}$$

Here the operators $L^{\pm}$ are defined by (1.2) and $M_m^{\pm}$, $H_m^{\pm}$ and Q_m are the mth Fourier components of the nonlinear terms $M^{\pm}$, $H^{\pm}$ and Q (see (1.2), (1.4) and (1.5) respectively). Throughout the subsequent analysis the superscript $\pm$ indicates an expression defined in $z \gtrless 0$.

For the Fourier component $m = 1$, it may be shown that $M_1^{\pm}$ are $O(\alpha^3)$. Now

$$L^{\pm}\left(-ikc+\epsilon\frac{\partial}{\partial T}, ik, \frac{\partial}{\partial z}+\epsilon\frac{\partial}{\partial Z}\right) w_1 = M_1^{\pm}, \tag{2.3}$$

and the appropriate solution for w_1 is thus

$$w_1 = \alpha A^{\pm}(T,Z)\exp\{\pm in^{\pm}z\}+\alpha_0 I^{\pm}(T,Z)\exp\{\mp in^{\pm}z\}+O(\alpha^5). \tag{2.4}$$

Here $n^{\pm}$ are given by (1.7). The error term is recorded as $O(\alpha^5)$, rather than $O(\alpha^3)$, in anticipation of the result that the leading term in $M_1^{\pm}$ is proportional to w_1. Indeed, it may be shown that

$$M_1^{\pm} = -2k^2(c\mp 1)^{-1}u_0 w_1+O(\alpha^5), \tag{2.5}$$

where u_0 is the mean horizontal velocity, and is $O(\alpha^2)$. Substituting (2.4) into (2.3), it follows that

$$\alpha\epsilon\frac{\partial A^\pm}{\partial Z} = \frac{\pm\alpha\epsilon}{kn^\pm(c\mp 1)^3}\frac{\partial A^\pm}{\partial T}+N_1^\pm, \tag{2.6a}$$

where

$$N_1^\pm = \frac{\pm\alpha i u_0 A^\pm}{n^\pm(c\mp 1)^3}+O(\alpha^5). \tag{2.6b}$$

Now the vertical group velocity for an internal gravity wave of vertical wavenumber n, horizontal wavenumber k and intrinsic frequency ω^* is

$$\partial\omega^*/\partial n = -n\omega^*/(n^2+k^2). \tag{2.7}$$

Indeed the dispersion relation for such a wave is $\omega^{*2}(n^2+k^2) = k^2$ [cf. (1.7)], from which (2.7) follows immediately. Substituting $n = \pm n^\pm$ and $\omega^* = k(c\mp 1)$, this vertical group velocity is $\mp kn^\pm(c\mp 1)^3$. Thus (2.6) shows that modulations in the amplitude $|A^\pm|$ propagate vertically upwards, with this group velocity, in $z>0$ (downwards in $z<0$), as the radiation condition (1.8) ensures that the group velocity has the appropriate sign. For if we put $A^\pm = R^\pm\exp\{i\phi^\pm\}$, then (2.6) shows that, since u_0 is *real*,

$$\frac{\partial R^\pm}{\partial Z} = \frac{\pm 1}{kn^\pm(c\mp 1)^3}\frac{\partial R^\pm}{\partial T}+O(\epsilon), \tag{2.8a}$$

$$\epsilon\frac{\partial \phi^\pm}{\partial Z} = \frac{\pm \epsilon}{kn^\pm(c\mp 1)^3}\frac{\partial \phi^\pm}{\partial T}\pm\frac{u_0}{n^\pm(c\mp 1)^3}+O(\epsilon^2). \tag{2.8b}$$

The nonlinear term $N_1^\pm$ affects only the phase of the wave. Further, (2.8*b*) shows that the total frequency $\overline{\omega}(\omega^*-ku_0-\epsilon\partial\phi/\partial T)$ is related to the total vertical wavenumber $\overline{n}(n+\epsilon\partial\phi/\partial z)$ by the dispersion relation for internal gravity waves, viz. $\overline{\omega}^2(\overline{n}^2+k^2) = k^2$. Since u_0 is proportional to $|A^\pm|^2$ [q.v. (3.18)], the only effect of the nonlinear term is to introduce an amplitude-dependent Doppler shift of ku_0. There is a similar equation to (2.6) for $I^\pm$ which shows that modulations in the incident wave propagate vertically downwards in $z>0$ (upwards in $z<0$). Equation (2.6) may be combined with (2.4) to show that

$$\frac{\partial w_1}{\partial z}+\epsilon\frac{\partial w_1}{\partial Z} = \{\pm i\tilde{n}^\pm\alpha A^\pm+N_1^\pm\}\exp\{\pm in^\pm z\}\mp i\tilde{n}^\pm\alpha_0 I^\pm\exp\{\mp in^\pm z\}+O(\alpha^5), \tag{2.9}$$

where $\tilde{n}^\pm$ are the *operators* defined by the symbols

$$\tilde{n}^\pm = n^\pm(\omega,k),\quad \omega = kc+i\epsilon\partial/\partial T, \tag{2.10}$$

and $n^\pm(kc,k)$ are the solutions of (1.7) which satisfy the radiation condition (1.8). Thus

$$\tilde{n}^\pm = n^\pm-\frac{i\epsilon}{kn^\pm(c\mp 1)^3}\frac{\partial}{\partial T}+O(\epsilon^2). \tag{2.11}$$

Next we obtain the counterparts of (2.4) for u_1 and p_1 and use the boundary conditions (2.2) with $m = 0$. The result is

$$\alpha\mathscr{D}(\omega,k)A = \alpha_0\{\mathscr{D}^+(\omega,k)I^\pm+\mathscr{D}^-(\omega,k)I^-\}+J, \tag{2.12}$$

where $I^\pm$ are evaluated at $z = 0\pm$,

$$\mathscr{D}(\omega,k) = -i(\omega-k)^2\tilde{n}^+-i(\omega+k)^2\tilde{n}^-, \tag{2.13a}$$

$$\mathscr{D}^\pm(\omega,k) = 2(\omega\mp k)\tilde{n}^\pm \tag{2.13b}$$

and J is the nonlinear expression

$$J = -k^2Q_1 - ik[F_{H1}]^{+}_{-} + (\omega - k)\,\tilde{n}^+H^+ + (\omega+k)\,\tilde{n}^-H^- + i(\omega-k)\,N_1^+ - i(\omega+k)\,N_1^- + O(\alpha^5). \qquad (2.14)$$

Here $N_1^{\pm}$ are evaluated at $z = 0\pm$. Equations (2.12)–(2.14) confirm the heuristic argument which led to (1.15). Now $\mathscr{D}(kc, k)$ vanishes because of the dispersion relation (1.10), so on expanding the operator ω [see (2.10)], we find that

$$i\alpha\epsilon\mathscr{D}_\omega(kc,k)\,\partial A/\partial T = \alpha_0\{\mathscr{D}^+(kc,k)\,I^+ + \mathscr{D}^-(kc,k)\,I^-\} + J + O(\alpha\epsilon^2, \alpha^5). \qquad (2.15)$$

Next, using (1.7) and (1.11), it may be shown that

$$\mathscr{D}_\omega(kc,k) = \begin{cases} 2ik(1-2k^2)(1-k^2)^{-\frac{1}{2}} & \text{for mode (i),} \\ -8ik^2c^2(1-c^2)^{-1} & \text{for mode (ii),} \end{cases} \qquad (2.16a)$$

and

$$\mathscr{D}^{\pm}(kc,k) = \begin{cases} -2k(1-k^2)^{\frac{1}{2}} & \text{for mode (i),} \\ -2k^2(1\pm c) & \text{for mode (ii).} \end{cases} \qquad (2.16b)$$

It remains to calculate the nonlinear term J. In order to do this, we must first calculate the second harmonics ($m = 2$) and the mean flow ($m = 0$). For the second harmonic, it transpires that $M_2^{\pm}$ are $O(\alpha^4)$ so the equation for w_2 is

$$L^{\pm}(-2ikc, 2ik, \partial/\partial z)\,w_2 = O(\alpha^4, \alpha^2|w_2|), \qquad (2.17)$$

where the terms $O(\alpha^2|w_2|)$ arise from the expansion of the operators $L^{\pm}$ with respect to ϵ ($= \alpha^2$). From (1.2) it follows that

$$\partial^2 w_2/\partial z^2 + (n_2^{\pm})^2 w_2 = O(\alpha^4, \alpha^2|w_2|), \qquad (2.18a)$$

where

$$(n_2^{\pm})^2 = (c \mp 1)^{-2} - 4k^2. \qquad (2.18b)$$

Using the dispersion relations (1.11), it follows that

$$(n_2^{\pm})^2 = \begin{cases} 1-4k^2 & \text{for mode (i),} \qquad (2.19a) \\ -2k^2(c\mp 1)^{-2}(c^2 \mp 4c + 1) & \text{for mode (ii).} \qquad (2.19b) \end{cases}$$

The appropriate solution of (2.18) is

$$w_2 = \alpha^2 A_2^{\pm}(T, Z)\exp\{\pm i n_2^{\pm} z\} + O(\alpha^4). \qquad (2.20)$$

Here we have inserted a factor α^2 in anticipation of the fact that the boundary conditions will show that w_2 is $O(\alpha^2)$. The solution (2.20) must satisfy a radiation condition; this is determined in a similar way to that which led to the radiation condition (1.8) for w_1, i.e. we let c have a small positive imaginary part and then require the solutions of (2.18) to decay exponentially as $|z| \to \infty$. Thus we shall require that

$$\operatorname{Im} n_2^{\pm} > 0, \quad \text{or} \quad \operatorname{Im} n_2^{\pm} = 0, \quad \operatorname{Re} n_2^{\pm} \gtrless 0. \qquad (2.21)$$

Here we have used the fact that when the dispersion relations (1.7) are satisfied $|c| < 1$. The solution w_2 is said to be radiating when $n_2^{\pm}$ is real and trapped when $n_2^{\pm}$ is imaginary. Thus mode (i) is radiating when $k < \frac{1}{2}$ and is trapped for $k > \frac{1}{2}$, while mode (ii) is radiating for $(2-3^{\frac{1}{2}}) \leqslant c < 1$ and $z > 0$, or $-1 < c \leqslant -(2-3^{\frac{1}{2}})$ and $z < 0$, and is trapped otherwise.

Using the boundary conditions for $m = 2$, it may be shown that (Grimshaw 1977) for mode (i)

$$A_2 = iA^2(1-k^2)^{\frac{1}{2}} + O(\alpha^2), \quad A_2^{\pm} = O(\alpha^2), \tag{2.22}$$

where we have relabelled $\zeta_2(T)$ as $\alpha^2 A_2(T)$. For mode (ii)

$$A_2 = ik\Delta^{-1}A^2\{-4k+(1-c^2)(n_2^+ + n_2^-)\} + O(\alpha^2), \tag{2.23a}$$

$$A_2^{\pm} = 8k^2c\Delta^{-1}A^2\{n_2^{\mp}(1 \pm c) \pm k(1 \mp c)\} + O(\alpha^2), \tag{2.23b}$$

where

$$\Delta = n_2^+(1-c)^2 + n_2^-(1+c)^2. \tag{2.23c}$$

We note that Δ is the counterpart for the second harmonic of the term (1.10) which produces the dispersion relation for the first harmonic. It is easily verified that, when $\frac{1}{2} < k < 2^{-\frac{1}{2}}$, Δ is not zero. However, for mode (i) and $k < \frac{1}{2}$, Δ is zero and the second harmonic is then resonant with the first harmonic; this situation has been examined fully elsewhere (Grimshaw 1977) and the results will be discussed briefly at the end of § 4.

The mean flow (i.e. the Fourier component $m = 0$) may be obtained by putting $m = 0$ in (2.2) or, alternatively, by averaging (1.1) over one wavelength with respect to x. This sort of calculation is now well understood, and has been described by Acheson (1976) for the case of mean flows forced by internal gravity waves (see also McIntyre & Weissman 1978). The result is that u_0, the mean horizontal velocity forced by the waves in $z \gtrless 0$, is

$$u_0 = 2\alpha^2 k^{-2}(c \mp 1)^{-3}|A^{\pm}|^2 + O(\alpha^4). \tag{2.24}$$

The mean vertical velocity is zero. It has been shown elsewhere (Grimshaw 1977) that the mean interface displacement ζ_0 is zero and that u_0, given by (2.24), and the corresponding mean pressure field will satisfy the boundary conditions (2.2) for $m = 0$. The role that u_0 plays in the total energy flux budget has been described by Acheson (1976) and McIntyre & Weissman (1978).

It remains to calculate the Fourier components $M_1^{\pm}$ [see (2.5)] and the various terms in J [see (2.14)]. This is a lengthy calculation involving the interaction of the Fourier component $m = 1$ with each of the Fourier components $m = 2$ and $m = 0$. It transpires that in (2.14) we may replace ω by kc and $\tilde{n}^{\pm}$ by $n^{\pm}$ to leading order, and that then J is proportional to $|A|^2 A$. The details are given elsewhere (Grimshaw 1977). The result is that

$$J = \frac{8ik^5c\alpha^3|A|^2A}{\Delta}\left\{-\frac{3(1+c^2)\Delta}{1-c^2} + 8kc - \frac{4c}{1-c^2}[n_2^+(1-c)^2 - n_2^-(1+c)^2]\right\} + O(\alpha^5). \tag{2.25}$$

3. Discussion of the amplitude equation

It was shown in § 1 that the amplitude equation is (1.17) (or (3.1) below). The coefficient β is found by first deriving the amplitude equation in the form (2.15), where $\mathscr{D}_\omega$ and $\mathscr{D}^{\pm}$ are given by (2.16), and then combining this with J as given by (2.25). The result is

$$\partial A/\partial T = \beta|A|^2 A + I, \tag{3.1}$$

where the coefficient β is given by

$$\beta = \begin{cases} 0 \quad \text{for mode (i)}, & (3.2a) \\ -\dfrac{3ik}{2c} + 4ik^3\left\{\dfrac{2k(1-c^2) - n_2^+(1-c)^2 + n_2^-(1+c)^2}{n_2^+(1-c)^2 + n_2^-(1+c)^2}\right\} \quad \text{for mode (ii)}, & (3.2b) \end{cases}$$

while I is given by
$$I = \beta^+ I^+ + \beta^- I^-, \tag{3.3a}$$

where
$$\beta^\pm = \begin{cases} (1-k^2)/(1-2k^2) \quad \text{for mode (i)}, & (3.3b) \\ -(1 \pm c)(1-c^2)/4c^2 \quad \text{for mode (ii)}. & (3.3c) \end{cases}$$

Here $I^\pm$ are evaluated at $z = 0\pm$, so I represents the effect of the incident wave packets on the interface, and is a known quantity. If the nonlinear effects are ignored then (3.1) implies that the growth of A is proportional to I, a result first obtained by McIntyre & Weissman (1978). Without loss of generality we shall assume that $I(T)$ is zero for $T < 0$, i.e. the incident wave packets arrive at the interface at $T = 0$.

For mode (i) (viz. $c = 0$ and $\frac{1}{2} < k < 2^{-\frac{1}{2}}$), β is zero and (3.1) may be integrated immediately. If I is zero (i.e. no incident wave packets), then (3.1) shows that A is a constant. Recalling the discussion at the end of § 1, this means that a time scale $O(\alpha^{-2})$ is not long enough to determine the evolution of A. If I is not zero, then

$$A = A_0 + \int_0^T I(T')\,dT', \tag{3.4}$$

where A_0 is the amplitude at $T = 0$ (if the sole excitation of the interface is due to the incident wave packets, then $A_0 = 0$). If I is constant, then (3.4) shows that A grows linearly with T. However, if the incident wave packets have finite extent ($I = 0$ for $T \geqslant T_0$ say), then

$$A \to A_0 + \int_0^\infty I(T')\,dT' \quad \text{as} \quad T \to \infty. \tag{3.5}$$

In this case then, the incident wave packet excites a steady (in the present frame of reference) periodic wave on the interface, which in turn radiates waves into $z \gtrless 0$. The magnitude of the excited wave is proportional both to the magnitude of the incident wave and to the length of the incident wave packet. In dimensional terms an incident wave packet of magnitude $\hat{I}^\pm$ (i.e. $UN^{-1}\alpha_0|I^\pm|$, where we recall that U is the velocity scale and N^{-1} is the time scale) and length (in time) Δt will excite an interfacial wave of amplitude $N\Delta t\beta^\pm\hat{I}^\pm$. Here $\beta^\pm$ are given by (3.3) and range from $\frac{3}{2}$ when k is $\frac{1}{2}$ to infinity when k is $2^{-\frac{1}{2}}$. If Δl is the length (in space) of the wave packet, then $\Delta l = |kn^\pm|\,\Delta t$ as $\mp kn^\pm$ are the vertical group velocities of the incident waves, and for a wave packet containing M wavelengths Δl is $2\pi M|n^\pm|^{-1}$. Hence we can estimate Δt to be $2\pi M\{k(1-k^2)\}^{-1}$. Lindzen (1974) discussed four observations of internal gravity waves associated with shear zones, and these observations are reproduced in table 1. Although these waves were observed in circumstances considerably more complex than the simple model considered here, we shall nevertheless use these observations to illustrate our theory. If we put $M = 10$, then the magnification factor $N\Delta t\beta^\pm$ is 9·5 for case 1 and 22·0 for case 2. For case 3 the observed wavenumber is in the unstable range, while for case 4 the observed wavenumbers straddle the critical wavenumber; for this last case the waves with wavelength 20 km have a magnification factor of 6·6 when $M = 10$.

Case	U ($\mathrm{m\,s^{-1}}$)	N ($\mathrm{s^{-1}}$)	UN^{-1} (m)	Observed wavelength (km)	Observed wavelength (units of UN^{-1})	Observed wavenumber k (units of NU^{-1})	Observed amplitude (m)	Observed amplitude (units of UN^{-1})
(1) Wallops Island, Virginia 7 February 1968 (Ottersten *et al.* 1973)	12	2×10^{-2}	600	6	10	0·63	125	0·21
(2) Haswell, Colorado 12 November 1971 (Hooke, Hall & Gossard 1973)	1	$2{\cdot}7 \times 10^{-2}$	37	0·35	9·45	0·66	7·5	0·20
(3) Wallops Island, Virginia 19 February 1970 (Hardy, Reed & Mather 1973)	7	2×10^{-2}	350	2·7	7·71	0·81	125	0·36
(4) Wallops Island, Virginia 18 March 1969 (Reed & Hardy 1972)	24	$1{\cdot}3 \times 10^{-2}$	1846	$15 \rightarrow 20$	$8{\cdot}1 \rightarrow 10{\cdot}8$	$0{\cdot}77 \rightarrow 0{\cdot}58$	450	0·24

TABLE 1. Observed amplitudes and wavelengths of internal gravity waves associated with shear zones.

For mode (ii) (viz. $2k^2(c^2+1) = 1$ and $\frac{1}{2} < k < 2^{-\frac{1}{2}}$) both the real and the imaginary part of β are non-zero. Let

$$\beta = \beta_R + i\beta_I. \tag{3.6}$$

Then we shall show that β_R is positive. Suppose first that $|c| < 2-3^{\frac{1}{2}}$; then the second harmonic (2.20) is trapped in both $z \gtrless 0$ and $n_2^{\pm} = im_2^{\pm}$, where $m_2^{\pm}$ are positive. In this case

$$\beta_R = \frac{8k^4(1-c^2)}{m_2^+(1-c)^2 + m_2^-(1+c)^2}, \quad |c| < 2-3^{\frac{1}{2}}, \tag{3.7}$$

and is clearly positive. If $2-3^{\frac{1}{2}} \leqslant c < 1$, then n_2^+ is real while $n_2^- = im_2^-$ (the second harmonic (2.20) is trapped in $z < 0$, but radiates in $z > 0$). In this case

$$\beta_R = \frac{2m_2^- k^3(1-c^2)}{3c}\{k(1+c)^2 - (1-c^2)n_2^+\}, \quad 2-3^{\frac{1}{2}} \leqslant c < 1, \tag{3.8}$$

and it may easily be shown that this is always positive. The case when

$$-1 < c \leqslant -(2-3^{\frac{1}{2}}),$$

$n_2^+ = im_2^+$ and n_2^- is real is similar to this last case. The fact that β_R is non-zero is due to both the asymmetry of the mode (i.e. $c \neq 0$) and the trapped nature of the second harmonic (either in $z \gtrless 0$, or in one of $z > 0$ or $z < 0$), as it is apparent from (3.2) that if $n_2^{\pm}$ were both real then β would be pure imaginary. Table 2 contains some calculated values of β for various values of c and k (for negative values of c, β_R has the same value while β_I is replaced by $-\beta_I$).

We have been unable to obtain the general solution of (3.1) when β is non-zero. The amplitude equation (3.1) is similar to amplitude equations encountered in the theory of stability of viscous flows (e.g. Landau & Lifshitz 1959, p. 104; or Stuart 1960); the principal difference is the absence of a linear term in A and the presence of the forcing term I. When I is zero, the solution is readily obtained, but in general (3.1) must be solved by numerical methods. Here we shall consider two extreme cases. For the first suppose that the time scale of the incident wave packet is very short, so that I may be approximated by $A_0\delta(T-T_0)$, where $\delta(T)$ is the delta-function. Then the solution of (3.1) is zero for $T < T_0$ and

$$A = A_0\{1-2\beta_R|A_0|^2(T-T_0)\}^{-\nu} \quad \text{for} \quad T \geqslant T_0, \tag{3.9a}$$

where

$$\nu = \beta(2\beta_R)^{-1}. \tag{3.9b}$$

It follows that, for $T \geqslant T_0$,

$$|A| = |A_0|\{1-2\beta_R|A_0|^2(T-T_0)\}^{-\frac{1}{2}}, \tag{3.10a}$$

$$\arg A = \arg A_0 - (\beta_I/2\beta_R)\ln\{1-2\beta_R|A_0|^2(T-T_0)\}, \tag{3.10b}$$

(3.9) is also the solution when I is zero, and $A = A_0$ at $T = T_0$. The solution (3.9) is thus also the asymptotic solution when an incident wave packet of finite extent encounters the interface (i.e. I is non-zero only for $0 < T < T_0$). The solution (3.9) develops a singularity in a time T_∞ (after T_0), where

$$T_\infty = (2\beta_R|A_0|^2)^{-1}. \tag{3.11}$$

As $(T-T_0) \to T_\infty$, both $|A|$ and $\arg A$ approach infinity, so A spirals outwards around the origin in the complex A plane. T_∞ may be regarded as a measure of the lifetime

c	k	β_R	β_I	β_I/β_R	$\beta_R\|A_0\|^2T_\infty$
0·1	0·704	0·971	−10·1	−10·4	0·408
0·2	0·693	0·896	−4·38	−4·89	0·475
0·3	0·677	0·512	−2·08	−4·07	0·512
0·4	0·657	0·185	−1·30	−7·00	0·427
0·5	0·632	0·0710	−0·874	−12·3	0·408
0·6	0·606	0·0252	−0·621	−24·6	0·419
0·7	0·579	0·00756	−0·463	−61·3	0·421
0·8	0·552	0·00162	−0·363	−224	0·421
0·9	0·526	0·000168	−0·296	−1760	0·421

TABLE 2. Calculated values of β_R, β_I and $\beta_R|A_0|^2T_\infty$ [see (3.16)] for mode (ii).

for periodic waves on the interface. In dimensional terms, if an incident wave packet excites waves on the interface of magnitude $\hat{A}$ (i.e. $UN^{-1}\alpha|A_0|$), then their estimated lifetime is $U^2(2\beta_R\hat{A}^2N^3)^{-1}$. For the observed waves tabulated in table 1, the estimated lifetime is 151 min for case 1 and 24 min for case 2; for the 20 km waves of case 4, the estimated lifetime is 24 h. (The value used for β_R is interpolated from table 2.) These times serve to illustrate the strong dependence of the estimated lifetime on the dimensionless wavelength, the lifetime increasing considerably as the wavelength increases.

Next suppose that the time scale of the incident wave is very long, so that we may put I equal to a constant; let $I = -\beta|A_0|^2A_0$. Then $A = A_0$ is an equilibrium solution; however, this equilibrium solution is unstable, and all solutions of (3.1) develop a singularity in a finite time. Let $A = A_0B$. Then

$$\partial B/\partial T = \beta|A_0|^2(|B|^2B-1). \tag{3.12}$$

It may be shown that this equation (two first-order equations for the real and imaginary parts of B) has just the one equilibrium point ($B \equiv 1$) and no limit cycles. Let

$$\psi = |B|^4-4B_R, \quad \text{where} \quad B = B_R+iB_I. \tag{3.13}$$

Then it is readily shown that

$$\partial\psi/\partial T = 4\beta_R|A_0|^2\{(B_R|B|^2-1)^2+B_I^2|B|^4\}. \tag{3.14}$$

The curves ψ = constant are the integral curves of (3.12) when $\beta_R = 0$. At the equilibrium point ($B \equiv 1$), ψ is -3 and the curves ψ = constant are approximately ellipses; as ψ increases to infinity, the curves ψ = constant more closely approximate circles. When β_R is positive, (3.14) shows that ψ increases with time T, and hence all solutions of (3.12) spiral outwards in the complex B plane. Further, when $|B|$ becomes sufficiently large, B will approach the asymptotic solution (3.9). Indeed, it may be shown that

$$B = B_0\{1-2\beta_R|A_0|^2|B_0|^2(T-T_0)\}^{-\nu}\{1+O[(T-T_0)^{\frac{1}{2}}|B_0|^{-3}]\} \tag{3.15}$$

for sufficiently large values of $|B_0|$; here B_0 is the value of B when T is T_0 and ν is defined by (3.9). It follows from (3.15) that B develops a singularity in a time T_∞, where

$$T_\infty = T_0+(2\beta_R|A_0|^2|B_0|^2)^{-1}. \tag{3.16}$$

The values of B_0 and T_0 depend on the initial conditions for (3.12).

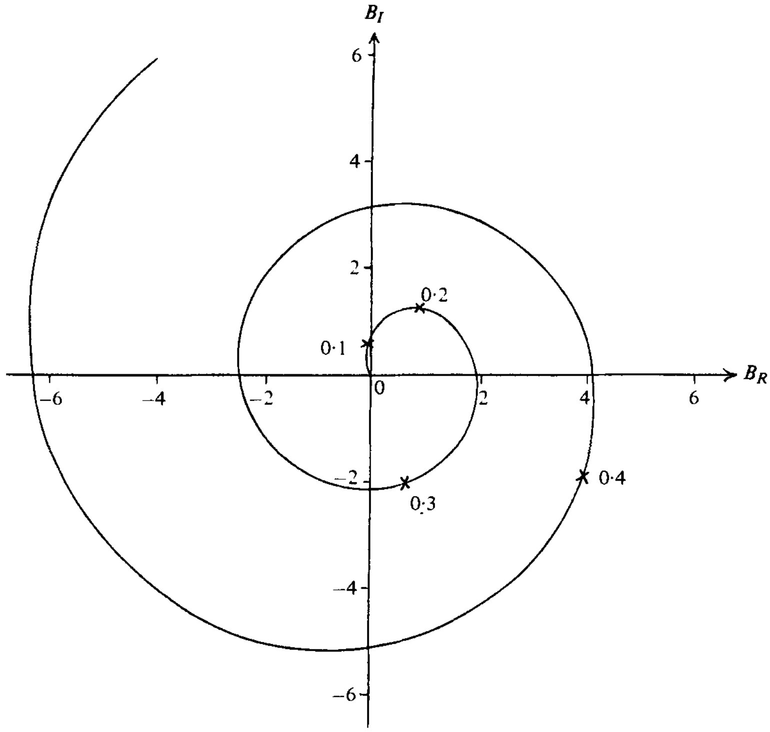

FIGURE 2. The computed evolution of B for mode (ii) and $c = 0{\cdot}4$. The crosses represent the values of B for $\beta_R|A_0|^2\,T$ equal to 0·1, 0·2, 0·3 and 0·4.

Equation (3.12) was integrated numerically for the initial condition $B = 0$ at $T = 0$ (i.e. the sole excitation is due to the incident wave packet); the integration was carried forward until (3.15) was satisfied, and so $|A_0|^2T_\infty$ could be determined. The results are displayed in table 2. The numerical results show that, as the ratio $|\beta_I/\beta_R| \to \infty$, $\beta_R|A_0|^2T_\infty$ approaches the asymptotic value 0·421. This result may be established analytically. For large values of the ratio $|\beta_I/\beta_R|$, (3.14) shows that ψ varies only slightly during one cycle around the origin, and each cycle around the origin follows an integral curve of ψ closely. Hence we may obtain an approximate solution by integrating (3.14) around an integral curve of ψ, and obtain

$$P(\psi)\frac{\partial\psi}{\partial T} \approx 4\beta_R|A_0|^2\oint_{\psi\,=\,\text{constant}}\{(B_R|B|^2-1)^2+|B|^4B_I^2\}\,dT, \tag{3.17}$$

where $P(\psi)$ is the time required to traverse one cycle on the integral curve ψ = constant (assuming $\beta_R \approx 0$). This determines $\partial\psi/\partial T$ as a function of ψ, and integration from ψ equals zero to infinity will determine $\beta_R|A_0|^2T_\infty$. This procedure also yields

$$\beta_R|A_0|^2T_\infty \simeq 0{\cdot}421$$

when $|\beta_I/\beta_R|$ is large. In figure 2 we show a plot of B as it evolves in time for $c = 0{\cdot}4$ (the results for other values of c are similar); it is apparent that, for most of the time up to T_∞, $|B|$ increases steadily while B travels around the origin a relatively small number of times. During the final passage to T_∞, B is governed by (3.15) and as $|B|$

increases B travels rapidly around the origin. T_∞ may be regarded as a measure of the lifetime of periodic waves on the interface. In dimensional terms, if the incident wave packet has amplitude $\hat{I}\pm$, then the estimated lifetime is $U^2N^{-3}\hat{A}^{-2}|A_0|^2T_\infty$, where $|\beta\pm\hat{I}\pm| = N^2U^{-2}|\beta\hat{A}^3|$. For the observed waves tabulated in table 1, the estimated lifetime is 144 min for case 1 and 27 min for case 2; for the 20 km waves of case 4 the estimated lifetime is 20 h. Here the observed amplitude has been used to determine $\hat{A}$ and hence infer $\hat{I}\pm$, while the value of $|A_0|^2T_\infty$ has been interpolated from table 2.

4. Summary

The principal purpose of this paper is to examine the weakly nonlinear aspects of resonant over-reflexion for the specific case of a Helmholtz velocity profile embedded in an infinite continuously stratified Boussinesq fluid. The range of dimensionless wavenumbers considered is $\frac{1}{2} < k < 2^{-\frac{1}{2}}$ (the length scale of the model is UN^{-1}). A heuristic argument, given at the end of § 1, suggested that the interface displacement $\alpha A(T)$, where $T = \alpha^2 t$, evolves on a time scale α^{-2}, where α is a small parameter measuring the magnitude of the interface displacement, and may be excited by an incident wave packet of magnitude $O(\alpha^3)$. It was suggested that the amplitude equation is (1.17), or (3.1), viz.

$$\partial A/\partial T = \beta|A|^2A + I, \tag{4.1}$$

where I is a known forcing term [q.v. (3.3)]. The subsequent analysis outlined in § 2 confirmed that this is the required amplitude equation, and enabled us to calculate the coefficient β [q.v. (3.2)]. Although our detailed analysis is confined to the specific case of a Helmholtz velocity profile, the heuristic argument at the end of § 1 suggests that the amplitude equation for resonant over-reflexion may take the same form as (4.1) in other situations. Once A has been determined from (4.1), the behaviour of $A^\pm$ in $z \gtrless 0$ is determined from (2.6), or (2.8); as described in § 2, these represent internal gravity waves propagating away from the interface.

The present choice of a vortex-sheet model was made solely for analytical convenience, although it suffers from the disadvantage that it possesses, for $k > 2^{-\frac{1}{2}}$, unstable linear modes and is too simple for the results to be applied directly to atmospheric observations. However, the use of a more sophisticated model with a continuous velocity profile and hence a shear layer of finite thickness is analytically much more difficult owing mainly to the singularity at the critical level which occurs in linear theory. Further, it was pointed out by Drazin & Howard (1966, p. 67) that the stability characteristics of a continuous velocity profile in the long-wavelength limit may not necessarily coincide with the stability characteristics of a vortex-sheet model. The reality of this difficulty was recently confirmed by Blumen, Drazin & Billings (1975) in an analysis of the linear stability of a shear layer in an inviscid compressible (unstratified) fluid; they found the existence of an unstable mode of long wavelength (with small growth rate) which exists for all values of the Mach number greater than 1, whereas the corresponding vortex-sheet model predicted stability for values of the Mach number greater than $2^{\frac{3}{2}}$. The possibility of this phenomenon occurring also for shear flows in inviscid stratified fluids is supported by the work of Einaudi & Lalas (1976), who calculated the linear stability properties of a continuous shear layer in a stratified fluid of bounded vertical extent; they established the existence of unstable modes of long wavelength and conjectured from their results that these unstable modes may continue to exist in an infinite fluid. On the other hand Eltayeb & McKenzie (1975) have established the existence of over-reflexion for internal gravity waves incident upon a finite shear layer, although they did not attempt to identify resonant over-reflexion or the stability characteristics of their model. In summary, it would seem that, while the use of a continuous velocity profile is desirable and more realistic, it is, at the present time, analytically intractable in relation to nonlinear effects. Whether or not an amplitude equation such as (4.1) can be found for other models where resonant over-reflexion occurs, the vortex-sheet model allows us to calculate the coefficient β explicitly and make some deductions about nonlinear effects.

For mode (i) [q.v. (1.11)] we find that β is zero [see (3.2)]. The evolution of A on the time scale $\alpha^2 t$ is completely determined by the forcing term I, provided that the incident wave is $O(\alpha^3)$; the effect of nonlinear terms requires the use of a longer time scale (say $\alpha^4 t$) than that considered in this paper. In the absence of the forcing term (i.e. when the excitation by the incident wave has ceased), this mode exists, in its linear form, in the weakly nonlinear regime, and the only nonlinear effect is a change in the vertical phase speed (q.v. (2.8) and the subsequent discussion). Thus the vanishing of the coefficient β establishes the existence of a finite amplitude steady wave on the interface, accompanied by internal gravity waves radiating away from the interface. Further, this steady wave can be excited by a transient incident wave. For the range of wavenumbers considered the interfacial wave can be ten or twenty times the magnitude of the incident wave.

For mode (ii) [q.v. (1.11)] we find that β_R, the real part of β, is positive (the imaginary part of β is also non-zero). The solutions of (4.1) are found in two cases, when I is a delta-function and when I is a constant (q.v. § 3); in general, I will lie somewhere between these extremes. In both cases it is found that A develops a singularity in a finite time. It seems likely that this will occur whenever the forcing term I is bounded. Hence we conclude that mode (ii) is unstable; the instability is due to the nonlinear self-interaction of the primary wave with its second harmonic (the self-interaction of the primary wave with the induced mean flow contributes only to the imaginary part of β). It is perhaps surprising that a completely neutral wave can undergo this form of nonlinear instability, as previous calculations of nonlinear effects for neutral waves in inviscid systems have produced a coefficient β with zero real part. The fact that β has a positive real part here is due to the fact that the system is unbounded in vertical extent, and the neutral wave is a resonant over-reflexion mode. Hence, while the vertical wavenumbers $n^{\pm}$ [q.v. (1.7) and (1.12)] for the primary mode are real, the corresponding vertical wavenumbers $n_2^{\pm}$ [q.v. (2.19)] for the second harmonic are not both real; it is this fact which causes the real part of β to be positive. We conjecture that this instability may occur in other systems exhibiting resonant over-reflexion. For the present case we calculated in § 3 the time from the initial excitation to the singularity for the two extreme cases when I is a delta-function and when I is a constant; in both cases we found that this time (dimensionless), interpreted as a lifetime for the mode (ii), is proportional to $(\beta_R|A_0|^2)^{-1}$, where $|A_0|$ is measure of the magnitude of the initial excitation. The coefficient β_R depends strongly on the dimensionless wavenumber k (table 2), decreasing as k decreases. We estimate lifetimes ranging from 20 min (for waves with dimensionless wavenumber close to the critical value $k_c = 2^{-\frac{1}{2}}$) to 20 h (dimensionless wavenumber close to $\frac{1}{2}$).

When $k < \frac{1}{2}$, the primary wave for mode (i) is in resonance with its second harmonic (i.e. the dispersion relation (1.10) is satisfied with $n^{\pm}$ replaced by $n_2^{\pm}$, the vertical wavenumbers for the second harmonic). This situation has been analysed in detail elsewhere (Grimshaw 1977). It has been shown that the resonance is weak; the second harmonic is $O(\alpha^2)$. The final result is that A is again given by (4.1) with β equal to zero, so the determination of A in terms of I is unchanged from the case $\frac{1}{2} < k < 2^{-\frac{1}{2}}$, the only change being in the expression for the second harmonic. One point of interest which arises is that when $k < \frac{1}{3}$ the third harmonic enters into a resonance with the first two harmonics (i.e. the dispersion relation (1.10) is satisfied with $n^{\pm}$ replaced by $n_3^{\pm}$, the appropriate vertical wavenumbers for the third harmonic). In general, for $k < 1/m$, the first m harmonics are in resonance and clearly, as $k \rightarrow 0$, the concept of a Fourier analysis, implied by (2.1), fails.

Finally, when k is close to the critical value k_c ($= 2^{-\frac{1}{2}}$), modes (i) and (ii) coalesce. The dispersion relation (1.10) then has three solutions; consequently $\mathscr{D}(\omega, k)$ has a third-order zero for ω, and the amplitude equation (1.15) is a third-order equation in time T. This case, in the absence of an incident wave packet, was examined by Grimshaw (1976); the effects of an incident wave packet have been described in detail by Grimshaw (1977). It was shown that, for a delta-function forcing ($I(T) = A_0\delta(T)$), $|A|$ undergoes a large amplitude oscillation, while, for continual constant forcing, $|A|$ grows indefinitely as $|T|^{\frac{1}{3}}$. By contrast with the behaviour of mode (ii) for $k < k_c$, it would seem that the ability of the three modes [mode (i) and the two modes of type (ii)] to interact is inhibiting in this circumstance, and prevents a singularity from developing in a finite time.

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